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# FORCING IN RAMSEY THEORY (Infinite Combinatorics and Forcing Theory)

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## FORCING IN RAMSEY THEORY

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**ABSTRACT.** Ramsey theory and forcing have a symbiotic relationship. At the RIMS Symposium on Infinite Combinatorics and Forcing Theory in 2016, the author gave three tutorials on *Ramsey theory in forcing*. The first two tutorials concentrated on forcings which contain dense subsets forming topological Ramsey spaces. These forcings motivated the development of new Ramsey theory, which then was applied to the generic ultrafilters to obtain the precise structure Rudin-Keisler and Tukey orders below such ultrafilters. The content of the first two tutorials has appeared in a previous paper [5]. The third tutorial concentrated on uses of forcing to prove Ramsey theorems for trees which are applied to determine big Ramsey degrees of homogeneous relational structures. This is the focus of this paper.

### 1. OVERVIEW OF TUTORIAL

Ramsey theory and forcing are deeply interconnected in a multitude of various ways. At the RIMS Conference on Infinite Combinatorics and Forcing Theory, the author gave a series of three tutorials on *Ramsey theory in forcing*. These tutorials focused on the following implications between forcing and Ramsey theory:

- (1) Forcings adding ultrafilters satisfying weak partition relations, which in turn motivate new topological Ramsey spaces and canonical equivalence relations on barriers;
- (2) Applications of Ramsey theory to obtain the precise Rudin-Keisler and Tukey structures below these forced ultrafilters;
- (3) Ramsey theory motivating new forcings and associated ultrafilters;
- (4) Forcing to obtain new Ramsey theorems for trees;
- (5) Applications of new Ramsey theorems for trees to obtain Ramsey theorems for homogeneous relational structures.

The first two tutorials focused on areas (1) - (3). We presented work from [10], [11] [9], [3], [4], and [2], in which dense subsets of forcings generating ultrafilters satisfying some weak partition properties were shown to form topological Ramsey spaces. Having obtained the canonical equivalence relations on fronts for these topological Ramsey spaces, they may be applied to obtain the precise initial Rudin-Keisler and Tukey structures. An exposition of this work has already appeared in [5].

The third tutorial concentrated on areas (4) and (5). We focused particularly on the Halpern-Läuchli Theorem and variations for strong trees. Extensions and analogues of this theorem have found applications in homogeneous relational structures. The majority of this article will concentrate on Ramsey theorems for trees due to (in historical order) Halpern-Läuchli [14]; Milliken [19]; Shelah [25]; Džamonja, Larson, and Mitchell [12]; Dobrinen and Hathaway [7]; Dobrinen [6]; and very recently Zhang [28]. These theorems have important applications to finding big Ramsey degrees for homogeneous structures.

We say that an infinite structure  $\mathcal{S}$  has *finite big Ramsey degrees* if for each finite substructure  $F$  of  $\mathcal{S}$  there is some finite number  $n(F)$  such that for any coloring of all copies of  $F$  in  $\mathcal{S}$  into finitely many colors, there is a substructure  $\mathcal{S}'$  if  $\mathcal{S}$  which is isomorphic to  $\mathcal{S}$  and such that all copies of  $F$  in  $\mathcal{S}'$  take no more than  $n(F)$  colors. The Halpern-Läuchli and Milliken Theorems, and other related Ramsey theorems on trees, have been instrumental in proving finite big Ramsey degrees for certain homogeneous relational structures. Section 2 contains Harrington's forcing proof of the Halpern-Läuchli Theorem.

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This is then applied to obtain Milliken's Theorem for strong subtrees. Applications of Milliken's theorem to obtain finite big Ramsey degrees are shown in Section 3. There, we provide the main ideas of how Sauer applied Milliken's Theorem to prove that the random graph on countably many vertices has finite big Ramsey degrees in [24]. Then we briefly cover applications to Devlin's work in [1] on finite subsets of the rationals, and Laver's work in [18] on finite products of the rationals.

In another vein, proving whether or not homogeneous relational structures omitting copies of a certain finite structure have finite big Ramsey degrees has been an elusive endeavor until recently. In [6], the author used forcing to prove the needed analogues of Milliken's Theorem and applied them to prove the universal triangle-free graph has finite big Ramsey degrees. The main ideas in that paper are covered in Section 4.

The final Section 5 addresses analogues of the Halpern-Läuchli Theorem for trees of uncountable height. The first such theorem, due to Shelah [25] (strengthened in [12]) considers finite antichains in one tree of measurable height. This was applied by Džamonja, Larson, and Mitchell to prove the consistency of a measurable cardinal  $\kappa$  and the analogue of Devlin's result for the  $\kappa$ -rationals in [12]; and that the  $\kappa$ -Rado graph has finite big Ramsey degrees in [13]. Recent work of Hathaway and the author in [7] considers more than one tree and implications for various uncountable cardinals. We conclude the paper with very recent results of Zhang in [28] obtaining the analogue of Laver's result at a measurable cardinal.

The existence of finite big Ramsey degrees has been of interest for some time to those studying homogeneous structures. In addition to those results considered in this paper, big Ramsey degrees have been investigated in the context of ultrametric spaces in [22]. A recent connection between finite big Ramsey degrees and topological dynamics has been made by Zucker in [29]. Any future progress on finite big Ramsey degrees will have implications for topological dynamics.

The author would like to thank Timothy Trujillo for creating most of the diagrams used in the tutorials, and the most complex ones included here.

## 2. THE HALPERN-LÄUCHLI AND MILLIKEN THEOREMS

Ramsey theory on trees is a powerful tool for investigations into several branches of mathematics. The Halpern-Läuchli Theorem was originally proved as a main technical lemma enabling a later proof of Halpern and Lévy that the Boolean prime ideal theorem is strictly weaker than the Axiom of Choice, assuming the ZF axioms (see [15]). Many variations of this theorem have been proved. We will concentrate on the strong tree version of the Halpern-Läuchli Theorem, referring the interested reader to Chapter 3 in [26] for a compendium of other variants. An extension due to Milliken, which is a Ramsey theorem on strong trees, has found numerous applications finding precise structural properties of homogeneous structures, such as the Rado graph and the rational numbers, in terms of Ramsey degrees for colorings of finite substructures. This will be presented in the second part of this section.

Several proofs of the Halpern-Läuchli Theorem are available in the literature. A proof using the technique of forcing was discovered by Harrington, and is regarded as providing the most insight. It was known to a handful of set theorists for several decades. His proof uses a which satisfies the following partition relation for colorings of subsets of size  $2d$  into countably many colors, and uses Cohen forcing to add many paths through each of the trees.

**Definition 1.** Given cardinals  $d, \sigma, \kappa, \lambda$ ,

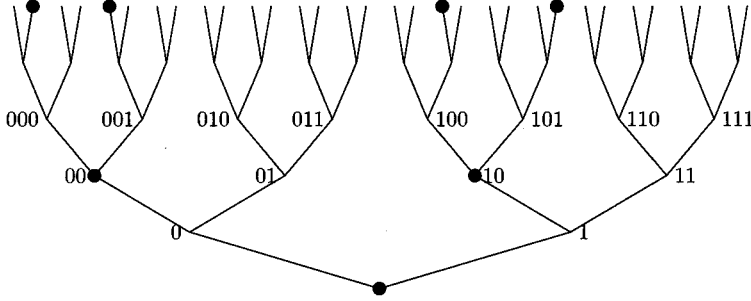
$$(1) \quad \lambda \rightarrow (\kappa)_\sigma^d$$

means that for each coloring of  $[\lambda]^d$  into  $\sigma$  many colors, there is a subset  $X$  of  $\lambda$  such that  $|X| = \kappa$  and all members of  $[X]^d$  have the same color.

The following is a ZFC result guaranteeing cardinals large enough to have the Ramsey property for colorings into infinitely many colors.

**Theorem 2** (Erdős-Rado). *For  $r < \omega$  and  $\mu$  an infinite cardinal,*

$$\beth_r(\mu)^+ \rightarrow (\mu^+)_\mu^{r+1}.$$

FIGURE 1. A strong tree isomorphic to  $2^{\leq 2}$ 

The book [27] of Farah and Todorcevic contains a forcing proof of the Halpern-Läuchli Theorem. The proof there is a modified version of Harrington's original proof. It uses a cardinal satisfying the weaker partition relation  $\kappa \rightarrow (\aleph_0)_2^d$ , which is satisfied by the cardinal  $\beth_{d-1}^+$ . This is important if one is interested in obtaining the theorem from weaker assumptions, and was instrumental in motivating the main result in [7] which is presented in Section 5. One could conceivably recover Harrington's original argument from Shelah's proof of the Halpern-Läuchli Theorem at a measurable cardinal (see [25]). However, his proof is more complex than simply lifting Harrington's argument to a measurable cardinal, as he obtains a stronger version, but only for one tree (see Section 5). Thus, we present here the simplest version of Harrington's forcing proof, filling a hole in the literature at present. This version was outlined to the author in 2011 by Richard Laver, and the author has filled in the gaps.

A *tree* on  $\omega^{<\omega}$  is a subset  $T \subseteq \omega^{<\omega}$  which is closed under meets. Thus, in this article, a tree is not necessarily closed under initial segments. We let  $\hat{T}$  denote the set of all initial segments of members of  $T$ ; thus,  $\hat{T} = \{s \in \omega^{<\omega} : \exists t \in T (s \subseteq t)\}$ . Given any tree  $T \subseteq \omega^{<\omega}$  and a node  $t \in T$ , let  $\text{spl}_T(t)$  denote the set of all immediate successors of  $t$  in  $\hat{T}$ ; thus,  $\text{spl}_T(t) = \{u \in \hat{T} : u \supseteq t \text{ and } |u| = |t| + 1\}$ . Notice that the nodes in  $\text{spl}_T(t)$  are not necessarily nodes in  $T$ . For a tree  $T \subseteq \omega^{<\omega}$  and  $n < \omega$ , let  $T(n)$  denote  $T \cap 2^n$ ; thus,  $T(n) = \{t \in T : |t| = n\}$ . A set  $X \subseteq T$  is a *level set* if all nodes in  $X$  have the same length. Thus,  $X \subseteq T$  is a level set if  $X \subseteq T(n)$  for some  $n < \omega$ .

Let  $T \subseteq \omega^{<\omega}$  be a finitely branching tree with no terminal nodes such that  $\hat{T} = T$ , and each node in  $T$  splits into at least two immediate successors. A subtree  $S \subseteq T$  is an infinite *strong subtree* of  $T$  if there is an infinite set  $L \subseteq \omega$  of levels such that

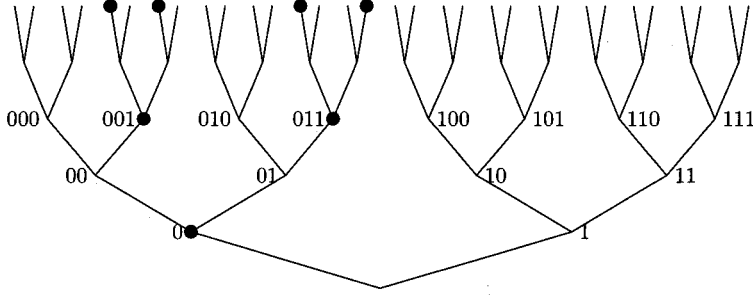
- (1)  $S = \bigcup_{l \in L} \{s \in S : |s| = l\}$ ;
- (2) for each node  $s \in S$ ,  $s$  splits in  $S$  if and only if  $|s| \in L$ ;
- (3) if  $|s| \in L$ , then  $\text{spl}_S(s) = \text{spl}_T(s)$ .

$S$  is a *finite strong subtree* of  $T$  if there is a finite set of levels  $L$  such that (1) holds, and every non-maximal node in  $S$  at a level in  $L$  splits maximally in  $T$ . See Figures 1 and 2 for examples of finite strong trees isomorphic to  $2^{\leq 2}$ , as determined by the darkened nodes.

We now present Harrington's proof of the Halpern-Läuchli Theorem, as outlined by Laver and filled in by the author. Although the proof uses the set-theoretic technique of forcing, the whole construction takes place in the original model of ZFC, not some generic extension. The forcing should be thought of as conducting an unbounded search for a finite object, namely the next level set where homogeneity is attained.

**Theorem 3 (Halpern-Läuchli).** *Let  $1 < d < \omega$  and let  $T_i \subseteq \omega^{<\omega}$  be finitely branching trees such that  $\hat{T}_i = T_i$ . Let*

$$(2) \quad c : \bigcup_{n < \omega} \prod_{i < d} T_i(n) \rightarrow 2$$

FIGURE 2. Another strong tree isomorphic to  $2^{\leq 2}$ 

be given. Then there is an infinite set of levels  $L \subseteq \omega$  and strong subtrees  $S_i \subseteq T_i$  each with branching nodes exactly at the levels in  $L$  such that  $c$  is monochromatic on

$$(3) \quad \bigcup_{n \in L} \prod_{i < d} S_i(n).$$

*Proof.* Let  $c : \bigcup_{n < \omega} \prod_{i < d} T_i(n) \rightarrow 2$  be given. Let  $\kappa = \beth_{2d}$ . The following forcing notion  $\mathbb{P}$  will add  $\kappa$  many paths through each  $T_i$ ,  $i < d$ .  $\mathbb{P}$  is the set of conditions  $p$  such that  $p$  is a function of the form

$$(4) \quad p : d \times \vec{\delta}_p \rightarrow \bigcup_{i < d} T_i(l_p)$$

where

- (i)  $\vec{\delta}_p \in [\kappa]^{<\omega}$ ;
- (ii)  $l_p < \omega$ ; and
- (iii) for each  $i < d$ ,  $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(l_p)$ .

The partial ordering on  $\mathbb{P}$  is simply inclusion:  $q \leq p$  if and only if  $l_q \geq l_p$ ,  $\vec{\delta}_q \supseteq \vec{\delta}_p$ , and for each  $(i, \delta) \in d \times \vec{\delta}_p$ ,  $q(i, \delta) \supseteq p(i, \delta)$ .

$\mathbb{P}$  adds  $\kappa$  branches through each of tree  $T_i$ ,  $i < d$ . For each  $i < d$  and  $\alpha < \kappa$ , let  $\dot{b}_{i,\alpha}$  denote the  $\alpha$ -th generic branch through  $T_i$ . Thus,

$$(5) \quad \dot{b}_{i,\alpha} = \{ \langle p(i, \alpha), p \rangle : p \in \mathbb{P}, \text{ and } (i, \alpha) \in \text{dom}(p) \}.$$

Note that for each  $p \in \mathbb{P}$  with  $(i, \alpha) \in \text{dom}(p)$ ,  $p$  forces that  $\dot{b}_{i,\alpha} \restriction l_p = p(i, \alpha)$ .

*Remark.*  $(\mathbb{P}, \leq)$  is just a fancy form of adding  $\kappa$  many Cohen reals, where we add  $d \times \kappa$  many Cohen reals, and the conditions are homogenized over the levels of trees in the ranges of conditions and over the finite set of ordinals indexing the generic branches.

Let  $\mathcal{U}$  be a  $\mathbb{P}$ -name for a non-principal ultrafilter on  $\omega$ . To ease notation, we shall write sets  $\{\alpha_i : i < d\}$  in  $[\kappa]^d$  as vectors  $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle$  in strictly increasing order. For  $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle \in [\kappa]^d$ , rather than writing out  $\langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}} \rangle$  each time we wish to refer to these generic branches, we shall simply

$$(6) \quad \text{let } \dot{b}_{\vec{\alpha}} \text{ denote } \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}} \rangle.$$

For any  $l < \omega$ ,

$$(7) \quad \text{let } \dot{b}_{\vec{\alpha}} \restriction l \text{ denote } \{ \dot{b}_{i,\alpha_i} \restriction l : i < d \}.$$

The goal now is to find infinite pairwise disjoint sets  $K'_i \subseteq \kappa$ ,  $i < d$ , and a set of conditions  $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i < d} K'_i\}$  which are compatible, have the same images in  $T$ , and such that for some fixed  $\varepsilon^*$ , for each  $\vec{\alpha} \in \prod_{i < d} K'_i$ ,  $p_{\vec{\alpha}}$  forces  $c(\dot{b}_{\vec{\alpha}} \restriction l) = \varepsilon^*$  for ultrafilter  $\mathcal{U}$  many  $l$ . Moreover, we will find nodes  $t_i^*$ ,  $i \leq d$ , such that for each  $\vec{\alpha} \in \prod_{i < d} K'_i$ ,  $p_{\vec{\alpha}}(i, \alpha_i) = t_i^*$ . These will serve as the basis for the process of building the strong subtrees  $S_i \subseteq T_i$  on which  $c$  is monochromatic.

For each  $\vec{\alpha} \in [\kappa]^d$ , choose a condition  $p_{\vec{\alpha}} \in \mathbb{P}$  such that

- (1)  $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ ;
- (2)  $p_{\vec{\alpha}} \Vdash$  “There is an  $\varepsilon \in 2$  such that  $c(\vec{b}_{\vec{\alpha}} \restriction l) = \varepsilon$  for  $\dot{U}$  many  $l$ ”;
- (3)  $p_{\vec{\alpha}}$  decides a value for  $\varepsilon$ , label it  $\varepsilon_{\vec{\alpha}}$ ; and
- (4)  $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$ .

Such conditions  $p_{\vec{\alpha}}$  may be obtained as follows. Given  $\vec{\alpha} \in [\kappa]^d$ , take  $p_{\vec{\alpha}}^1$  to be any condition such that  $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}^1}$ . Since  $\mathbb{P}$  forces  $\dot{U}$  to be an ultrafilter on  $\omega$ , there is a condition  $p_{\vec{\alpha}}^2 \leq p_{\vec{\alpha}}^1$  such that  $p_{\vec{\alpha}}^2$  forces that  $c(\vec{b}_{\vec{\alpha}} \restriction l)$  is the same color for  $\dot{U}$  many  $l$ . Furthermore, there must be a stronger condition deciding which of the colors  $c(\vec{b}_{\vec{\alpha}} \restriction l)$  takes on  $\dot{U}$  many levels  $l$ . Let  $p_{\vec{\alpha}}^3 \leq p_{\vec{\alpha}}^2$  be a condition which decides this color, and let  $\varepsilon_{\vec{\alpha}}$  denote that color. Finally, since  $p_{\vec{\alpha}}^3$  forces that for  $\dot{U}$  many  $l$  the color  $c(\vec{b}_{\vec{\alpha}} \restriction l)$  will equal  $\varepsilon_{\vec{\alpha}}$ , there is some  $p_{\vec{\alpha}}^4 \leq p_{\vec{\alpha}}^3$  which decides some level  $l$  so that  $c(\vec{b}_{\vec{\alpha}} \restriction l) = \varepsilon_{\vec{\alpha}}$ . If  $l_{p_{\vec{\alpha}}^4} < l$ , let  $p_{\vec{\alpha}}$  be any member of  $\mathbb{P}$  such that  $p_{\vec{\alpha}} \leq p_{\vec{\alpha}}^4$  and  $l_{p_{\vec{\alpha}}} = l$ . If  $l_{p_{\vec{\alpha}}^4} \geq l$ , let  $p_{\vec{\alpha}} = \{((i, \delta), p_{\vec{\alpha}}^4(i, \delta) \restriction l) : (i, \delta) \in d \times \vec{\delta}_{p_{\vec{\alpha}}^4}\}$ , the truncation of  $p_{\vec{\alpha}}^4$  to images that have length  $l$ . The  $p_{\vec{\alpha}}$  forces that  $\vec{b}_{\vec{\alpha}} \restriction l = \{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}$ , and hence  $p_{\vec{\alpha}}$  forces that  $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$ .

We are assuming  $\kappa = \beth_{2d}$ , which is at least  $\beth_{2d-1}(\aleph_0)^+$ , so  $\kappa \rightarrow (\aleph_1)_{\aleph_0}^{2d}$  by Theorem 2.

Now we prepare for an application of the Erdős-Rado Theorem. Given two sets of ordinals  $J, K$  we shall write  $J < K$  if and only if every member of  $J$  is less than every member of  $K$ . Let  $D_e = \{0, 2, \dots, 2d-2\}$  and  $D_o = \{1, 3, \dots, 2d-1\}$ , the sets of even and odd integers less than  $2d$ , respectively. Let  $\mathcal{I}$  denote the collection of all functions  $\iota : 2d \rightarrow 2d$  such that  $\iota \restriction D_e$  and  $\iota \restriction D_o$  are strictly increasing sequences and

$$(8) \quad \{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \dots < \{\iota(2d-2), \iota(2d-1)\}.$$

Thus, each  $\iota$  codes two strictly increasing sequences  $\iota \restriction D_e$  and  $\iota \restriction D_o$ , each of length  $d$ . For  $\vec{\theta} \in [\kappa]^{2d}$ ,  $\iota(\vec{\theta})$  determines the pair of sequences of ordinals

$$(9) \quad (\theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d-2)}), (\theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d-1)}),$$

both of which are members of  $[\kappa]^d$ . Denote these as  $\iota_e(\vec{\theta})$  and  $\iota_o(\vec{\theta})$ , respectively. To ease notation, let  $\vec{\delta}_{\vec{\alpha}}$  denote  $\vec{\delta}_{p_{\vec{\alpha}}}$ ,  $k_{\vec{\alpha}}$  denote  $|\vec{\delta}_{\vec{\alpha}}|$ , and let  $l_{\vec{\alpha}}$  denote  $l_{p_{\vec{\alpha}}}$ . Let  $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$  denote the enumeration of  $\vec{\delta}_{\vec{\alpha}}$  in increasing order.

Define a coloring  $f$  on  $[\kappa]^{2d}$  into countably many colors as follows: Given  $\vec{\theta} \in [\kappa]^{2d}$  and  $\iota \in \mathcal{I}$ , to reduce the number of subscripts, letting  $\vec{\alpha}$  denote  $\iota_e(\vec{\theta})$  and  $\vec{\beta}$  denote  $\iota_o(\vec{\theta})$ , define

$$(10) \quad f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \langle \langle i, j \rangle : i < d, j < k_{\vec{\alpha}}, \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \langle \langle j, k \rangle : j < k_{\vec{\alpha}}, k < k_{\vec{\beta}}, \delta_{\vec{\beta}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle.$$

Let  $f(\vec{\theta})$  be the sequence  $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$ , where  $\mathcal{I}$  is given some fixed ordering. Since the range of  $f$  is countable, applying the Erdős-Rado Theorem, we obtain a subset  $K \subseteq \kappa$  of cardinality  $\aleph_1$  which is homogeneous for  $f$ . Take  $K' \subseteq K$  such that between each two members of  $K'$  there is a member of  $K$  and  $\min(K') > \min(K)$ . Take subsets  $K'_i \subseteq K'$  such that  $K'_0 < \dots < K'_{d-1}$  and each  $|K'_i| = \aleph_0$ .

**Claim 1.** *There are  $\varepsilon^* \in 2$ ,  $k^* \in \omega$ , and  $\langle t_{i,j} : j < k^* \rangle$ ,  $i < d$ , such that  $\varepsilon_{\vec{\alpha}} = \varepsilon^*$ ,  $k_{\vec{\alpha}} = k^*$ , and  $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle$ , for each  $i < d$ , for all  $\vec{\alpha} \in \prod_{i < d} K'_i$ .*

*Proof.* Let  $\iota$  be the member in  $\mathcal{I}$  which is the identity function on  $2d$ . For any pair  $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$ , there are  $\vec{\theta}, \vec{\theta}' \in [\kappa]^{2d}$  such that  $\vec{\alpha} = \iota_e(\vec{\theta})$  and  $\vec{\beta} = \iota_e(\vec{\theta}')$ . Since  $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta}')$ , it follows that  $\varepsilon_{\vec{\alpha}} = \varepsilon_{\vec{\beta}}$ ,  $k_{\vec{\alpha}} = k_{\vec{\beta}}$ , and  $\langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle = \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle$ .  $\square$

Let  $l^*$  denote the length of the nodes  $t_{i,j}$ .

**Claim 2.** *Given any  $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$ , if  $j, j' < k^*$  and  $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ , then  $j = j'$ .*

*Proof.* Let  $\vec{\alpha}, \vec{\beta}$  be members of  $\prod_{i < d} K'_i$  and suppose that  $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$  for some  $j, j' < k^*$ . For each  $i < d$ , let  $\rho_i$  be the relation from among  $\{<, =, >\}$  such that  $\alpha_i \rho_i \beta_i$ . Let  $\iota$  be the member of  $\mathcal{I}$  such that for each  $\vec{\gamma} \in [K]^d$  and each  $i < d$ ,  $\theta_{\iota(2i)} \rho_i \theta_{\iota(2i+1)}$ . Then there is a  $\vec{\theta} \in [K']^{2d}$  such that  $\iota_e(\vec{\theta}) = \vec{\alpha}$  and  $\iota_o(\vec{\theta}) = \vec{\beta}$ . Since between any two members of  $K'$  there is a member of  $K$ , there is a  $\vec{\gamma} \in [K]^d$  such that for each  $i < d$ ,  $\alpha_i \rho_i \gamma_i$  and  $\gamma_i \rho_i \beta_i$ , and furthermore, for each  $i < d-1$ ,  $\{\alpha_i, \beta_i < \gamma_i\} \subset \{\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}\}$ . Given that  $\alpha_i \rho_i \gamma_i$  and  $\gamma_i \rho_i \beta_i$  for each  $i < d$ , there are  $\vec{\mu}, \vec{\nu} \in [K]^{2d}$  such that  $\iota_e(\vec{\mu}) = \vec{\alpha}$ ,  $\iota_o(\vec{\mu}) = \vec{\gamma}$ ,  $\iota_e(\vec{\nu}) = \vec{\gamma}$ , and  $\iota_o(\vec{\nu}) = \vec{\beta}$ . Since  $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$ , the pair  $\langle j, j' \rangle$  is in the last sequence in  $f(\iota, \vec{\theta})$ . Since  $f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$ , also  $\langle j, j' \rangle$  is in the last sequence in  $f(\iota, \vec{\mu})$  and  $f(\iota, \vec{\nu})$ . It follows that  $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\gamma}}(j)$  and  $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\beta}}(j')$ . Hence,  $\delta_{\vec{\gamma}}(j) = \delta_{\vec{\gamma}}(j')$ , and therefore  $j$  must equal  $j'$ .  $\square$

For any  $\vec{\alpha} \in \prod_{i < d} K'_i$  and any  $\iota \in \mathcal{I}$ , there is a  $\vec{\theta} \in [K]^{2d}$  such that  $\vec{\alpha} = \iota_o(\vec{\theta})$ . By homogeneity of  $f$  and by the first sequence in the second line of equation (10), there is a strictly increasing sequence  $\langle j_i : i < d \rangle$  of members of  $k^*$  such that for each  $\vec{\alpha} \in \prod_{i < d} K'_i$ ,  $\vec{\delta}_{\vec{\alpha}}(j_i) = \alpha_i$ . For each  $i < d$ , let  $t_{i,j_i}^*$  denote  $t_{i,j_i}$ . Then for each  $i < d$  and each  $\vec{\alpha} \in \prod_{i < d} K'_i$ ,

$$(11) \quad p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \vec{\delta}_{\vec{\alpha}}(j_i)) = t_{i,j_i} = t_{i,j_i}^*.$$

**Lemma 4.** *The set of conditions  $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i < d} K'_i\}$  is compatible.*

*Proof.* Suppose toward a contradiction that there are  $\vec{\alpha}, \vec{\beta} \in \prod_{i < d} K'_i$  such that  $p_{\vec{\alpha}}$  and  $p_{\vec{\beta}}$  are incompatible. By Claim 1, for each  $i < d$  and  $j < k^*$ ,

$$(12) \quad p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)).$$

Thus, the only way  $p_{\vec{\alpha}}$  and  $p_{\vec{\beta}}$  can be incompatible is if there are  $i < d$  and  $j, j' < k^*$  such that  $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(j')$  but  $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) \neq p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j'))$ . Since  $p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j}$  and  $p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j')) = t_{i,j'}$ , this would imply  $j \neq j'$ . But by Claim 2,  $j \neq j'$  implies that  $\delta_{\vec{\alpha}}(j) \neq \delta_{\vec{\beta}}(j')$ , a contradiction. Therefore,  $p_{\vec{\alpha}}$  and  $p_{\vec{\beta}}$  must be compatible.  $\square$

To build the strong subtrees  $S_i \subseteq T_i$ , for each  $i < d$ , let  $\text{stem}(S_i) = t_i^*$ . Let  $l_0$  be the length of the  $t_i^*$ .

**Induction Assumption:** Assume  $1 \leq m < \omega$  and there are levels  $l_0, \dots, l_{m-1}$  and we have constructed finite strong subtrees  $S_i \upharpoonright (l_{m-1} + 1)$  of  $T_i$ ,  $i < d$ , such that for each  $j < m$ ,  $c$  takes color  $\varepsilon^*$  on each member of  $\prod_{i < d} S_i(l_j)$ .

For each  $i < d$ , let  $X_i$  denote the set of immediate extensions in  $T_i$  of the maximal nodes of  $S_i$ . For each  $i < d$ , let  $J_i$  be a subset of  $K'_i$  with the same size as  $X_i$ . For each  $i < d$ , label the nodes in  $X_i$  as  $\{q(i, \delta) : \delta \in J_i\}$ . Let  $\vec{J}$  denote  $\prod_{i < d} J_i$ . Notice that for each  $\vec{\alpha} \in \vec{J}$  and  $i < d$ ,  $q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i)$ .

We now construct a condition  $q \in \mathbb{P}$  such that for each  $\vec{\alpha} \in \vec{J}$ ,  $q \leq p_{\vec{\alpha}}$ . Let  $\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ . For each pair  $(i, \gamma)$  with  $i < d$  and  $\gamma \in \vec{\delta}_q \setminus J_i$ , there is at least one  $\vec{\alpha} \in \vec{J}$  and some  $j' < k^*$  such that  $\vec{\delta}_{\vec{\alpha}}(j') = \gamma$ . For any other  $\vec{\beta} \in \vec{J}$  for which  $\gamma \in \vec{\delta}_{\vec{\beta}}$ , since the set  $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$  is pairwise compatible by Lemma 4, it follows that  $p_{\vec{\beta}}(i, \gamma)$  must equal  $p_{\vec{\alpha}}(i, \gamma)$ , which is exactly  $t_{i,j'}^*$ . Further note that in this case,  $\vec{\delta}_{\vec{\beta}}(j'')$  must also equal  $\gamma$ : If  $j'' < k^*$  is the integer satisfying  $\gamma = \vec{\delta}_{\vec{\beta}}(j'')$ , then  $t_{i,j''}^* = p_{\vec{\beta}}(i, \vec{\delta}_{\vec{\beta}}(j'')) = p_{\vec{\beta}}(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t_{i,j'}^*$ , and hence  $j'' = j'$ . Let  $q(i, \gamma)$  be the leftmost extension of  $t_{i,j'}^*$  in  $T$ . By the above argument,  $q(i, \gamma)$  is well-defined.

Thus,  $q(i, \gamma)$  is defined for each pair  $(i, \gamma) \in d \times \vec{\delta}_q$ . Define

$$(13) \quad q = \{ \langle (i, \delta), q(i, \delta) \rangle : i < d, \delta \in \vec{\delta}_q \}.$$

**Claim 3.** *For each  $\vec{\alpha} \in \vec{J}$ ,  $q \leq p_{\vec{\alpha}}$ .*

*Proof.* Given  $\vec{\alpha} \in \vec{J}$ , by our construction for each pair  $(i, \gamma) \in d \times \vec{\delta}_{\vec{\alpha}}$ , we have  $q(i, \gamma) \supseteq p_{\vec{\alpha}}(i, \gamma)$ .  $\square$

To construct the  $m$ -th level of the strong trees  $S_i$ , take an  $r \leq q$  in  $\mathbb{P}$  which decides some  $l_m \geq l_q$  for which  $c(b_{\bar{\alpha}} \upharpoonright l_m) = \varepsilon^*$ , for all  $\bar{\alpha} \in \bar{J}$ . By extending or truncating  $r$ , we may assume, without loss of generality, that  $l_m$  is equal to the length of the nodes in the image of  $r$ . Notice that since  $r$  forces  $b_{\bar{\alpha}} \upharpoonright l_m = \{r(i, \alpha_i) : i < d\}$  for each  $\bar{\alpha} \in \bar{J}$ , and since the coloring  $c$  is defined in the ground model, it is simply true in the ground model that  $c(\{r(i, \alpha_i) : i < d\}) = \varepsilon^*$  for each  $\bar{\alpha} \in \bar{J}$ . For each  $i < d$  and  $\alpha_i \in J_i$ , extend the nodes in  $X_i$  to level  $l_m$  by extending  $q(i, \delta)$  to  $r(i, \delta)$ . Thus, for each  $i < d$ , we define  $S_i(l_m) = \{r(i, \delta) : \delta \in J_i\}$ . It follows that  $c$  takes value  $\varepsilon^*$  on each member of  $\prod_{i < d} S_i(l_m)$ .

For each  $i < d$ , let  $S_i = \bigcup_{m < \omega} S_i(l_m)$ , and let  $L = \{l_m : m < \omega\}$ . Then each  $S_i$  is a strong subtree of  $T_i$ , and  $c$  takes value  $\varepsilon^*$  on  $\bigcup_{l \in L} \prod_{i < d} S_i(l)$ .  $\square$

The Halpern-Läuchli Theorem is used to obtain a space of strong trees with important Ramsey properties.

**Definition 5** (Milliken space). The *Milliken space* is the triple  $(\mathcal{M}, \leq, r)$ , where  $\mathcal{M}$  consists of all infinite strong subtrees  $T \subseteq 2^{<\omega}$ ;  $\leq$  is the partial ordering defined by  $S \leq T$  if and only if  $S$  is a subtree of  $T$ ; and  $r_k(T)$  is the  $k$ -th restriction of  $T$ , meaning the set of all  $t \in T$  with  $< k$  splitting nodes in  $T$  below  $t$ .

Thus,  $r_k(T)$  is a finite strong tree with  $k$  many levels. Let  $\mathcal{AM}_k$  denote the set of all strong trees with  $k$  levels, and let  $\mathcal{AM}$  denote all finite strong trees. A topology on  $\mathcal{M}$  is generated by basic open sets of the form

$$(14) \quad [U, T] = \{S \in \mathcal{M} : \exists k (r_k(S) = U) \text{ and } S \leq T\},$$

where  $U \in \mathcal{A}$  and  $T \in \mathcal{M}$ . Milliken proved in [19] that, in current terminology, the space of all strong trees forms a topological Ramsey space. The properties of meager and having the property of Baire are defined in the standard way from the topology. A subset  $\mathcal{X} \subseteq \mathcal{M}$  is *Ramsey* if for every  $\emptyset \neq [U, T]$  there is an  $S \in [U, T]$  such that either  $[U, S] \subseteq \mathcal{X}$  or else  $[U, S] \cap \mathcal{X} = \emptyset$ .  $\mathcal{X}$  is *Ramsey null* if the second option occurs for any given  $[U, S]$ .

**Definition 6** ([26]). A triple  $(\mathcal{R}, \leq, r)$  is a *topological Ramsey space* if every subset of  $\mathcal{R}$  with the property of Baire is Ramsey and if every meager subset of  $\mathcal{R}$  is Ramsey null.

The following theorem is a consequence of Milliken's result that the space of all strong subtrees of  $2^{<\omega}$  is a topological Ramsey space. This theorem provides a powerful tool for obtaining finite big Ramsey degrees for the Rado graph and the rationals, considered in the next section.

**Theorem 7** (Milliken, [19]). *For each  $k < \omega$ ,  $T \in \mathcal{M}$ , and coloring of all finite strong subtrees of  $T$  with  $k$  levels, there is an infinite strong subtree  $S \leq T$  such that all finite strong subtrees of  $S$  with  $k$  levels have the same color.*

In the setting of topological Ramsey spaces, the following is the pigeonhole principle (**Axiom A.4** in [26]); it follows from an application of the Halpern-Läuchli Theorem as shown below. For  $U \in \mathcal{AM}_k$ ,  $r_{k+1}[U, T]$  denotes the set  $\{r_{k+1}(S) : S \in [U, T]\}$ .

**Lemma 8.** *Let  $k < \omega$ ,  $U \in \mathcal{AM}_k$ , and  $T \in \mathcal{M}$  such that  $r_{k+1}[U, T]$  is nonempty. Then for each coloring of  $r_{k+1}[U, T]$  into finitely many colors, there is an  $S \in [U, T]$  such that all members of  $r_{k+1}[U, S]$  have the same color.*

*Proof.* By induction on the number of colors, it suffices to consider colorings into two colors. Let  $c : r_{k+1}[U, T] \rightarrow 2$  be given. If  $k = 0$ , then  $r_1[U, T]$  is simply the set of nodes in  $T$ . In this case, the pigeonhole principle is exactly the Halpern-Läuchli Theorem on one tree. Now suppose  $k \geq 1$ . Note that there are  $2^k$  many immediate successors of the maximal nodes in  $U$ ; list these as  $s_i$ ,  $i < 2^k$ , and let  $T_i = \{t \in T : t \geq s_i\}$ . Let  $L$  denote the levels of the trees  $T_i$ ; that is,  $L$  is the set of all the lengths of the nodes in  $T_i$  which split. This set  $L$  is the same for each  $i < 2^k$ , since the  $T_i$  are cones in the strong tree  $T$  starting at the level one above the maximum lengths of nodes in  $U$ . Notice that  $r_{k+1}[U, T]$  is exactly the set of all  $U \cup \{u_i : i < 2^k\}$ , where  $\{u_i : i < 2^k\}$  is a member of  $\prod_{i < 2^k} T_i(l)$  for some  $l \in L$ . Let  $d$  be the coloring on  $\bigcup_{l \in L} \prod_{i < 2^k} T_i(l)$  induced by  $c$  as follows:

$$(15) \quad d(\langle u_i : i < 2^k \rangle) = c(U \cup \{u_i : i < 2^k\}).$$



Apply the Halpern-Läuchli Theorem for  $2^k$  many trees to obtain an infinite set of levels  $K \subseteq L$  and strong subtrees  $S_i \subseteq T_i$  with nodes at the levels in  $K$  such that  $d$  is monochromatic on  $\bigcup_{l \in K} \prod_{i < 2^k} S_i(l)$ . Let

$$(16) \quad S = U \cup \bigcup_{i < 2^k} S_i.$$

Then  $S$  is a strong subtree of  $T$  such that  $r_k(S) = U$  and  $c$  is monochromatic on  $r_{k+1}[U, S]$ .  $\square$

Theorem 7 is obtained by Lemma 8 using induction on  $k$ .

### 3. APPLICATIONS OF MILLIKEN'S THEOREM TO HOMOGENEOUS BINARY RELATIONAL STRUCTURES

The *random graph* is the graph on  $\omega$  many vertices such that given any two vertices, there is a 50% chance that there is an edge between them. A graph  $R$  on  $\omega$  many vertices is random if and only if it is universal for all countable graphs; that is, every countable graph embeds into  $R$ . This is equivalent to  $R$  being homogeneous; any isomorphism between two finite subgraphs of  $R$  can be extended to an automorphism of  $R$ . Another characterization of the random graph is that it is the Fraïssé limit of the Fraïssé class of finite graphs. As the random graph on  $\omega$  many vertices was first constructed by R. Rado, we will call it the *Rado graph*, and denote it by  $\mathcal{R}$ .

The Rado graph has the Ramsey Property for vertex colorings.

**Fact 9** (Folklore). *For each coloring of the vertices of the Rado graph  $\mathcal{R}$  into finitely many colors, there is a subgraph  $\mathcal{R}'$  which is also a Rado graph, in which the vertices are homogeneous for  $c$ .*

For finite colorings of the copies of a graph with more than one vertex, it is not always possible to cut down to one color in a copy of the full Rado graph. However, Sauer showed that there is a bound on the number of colors that cannot be avoided.

**Theorem 10** (Sauer, [24]). *The Rado graph has finite big Ramsey degrees.*

The following outlines the key steps in Sauer's proof: Let  $G$  be a finite graph.

- (1) Trees can code graphs.
- (2) There are only finitely many isomorphism types of trees coding  $G$ , and only the strongly diagonal types matter.
- (3) For each isomorphism type of tree coding  $G$ , there is a way of enveloping it into a finite strong tree.
- (4) The coloring on copies of  $G$  can be transferred to color finite strong trees, and Milliken's Theorem may be applied to these 'strong tree envelopes' of the trees coding  $G$ .
- (5) Conclude that there is an infinite strong subtree of  $2^{<\omega}$  which includes a code of  $\mathcal{R}$  and on which there is one color per isomorphism type of tree coding  $G$ .

Let  $s$  and  $t$  be nodes in  $2^{<\omega}$ , and suppose  $|s| < |t|$ . If  $s$  and  $t$  represent vertices  $v$  and  $w$ , then  $s$  and  $t$  represent an edge between  $v$  and  $w$  if and only if  $t(|s|) = 1$ . Thus, if  $t(|s|) = 0$ , then  $s$  and  $t$  represent no edge between vertices  $v$  and  $w$ .

Let  $G$  be a graph. Enumerate the vertices of  $G$  in any order as  $\langle v_n : n < N \rangle$ , where  $N = |G|$ . Any collection of nodes  $\langle t_n : n < N \rangle$  in  $2^{<\omega}$  for which the following hold is a *tree coding*  $G$ : For each pair  $m < n < N$ ,

- (1)  $|t_m| < |t_n|$ ; and
- (2)  $t_n(|t_m|) = 1 \Leftrightarrow v_n E v_m$ .

The integer  $t_n(|t_m|)$  is called the *passing number* of  $t_n$  at  $t_m$ .

A tree  $Z \subseteq 2^{<\omega}$  is *strongly diagonal* if  $Z$  is the meet closure of its terminal nodes, no two terminal nodes of  $Z$  have the same length, for each  $l < \omega$ , there is at most one splitting node or terminal node in  $Z$  of length  $l$ , and at a splitting node, all other nodes not splitting at that level have passing number 0.

**Definition 11** (Sauer, [24]). Let  $S$  and  $T$  be subtrees of  $2^{<\omega}$ . A function  $f : S \rightarrow T$  is a *strong similarity* of  $S$  to  $T$  if for all nodes  $s, t, u, v \in S$ ,

- (1)  $f$  is a bijection.

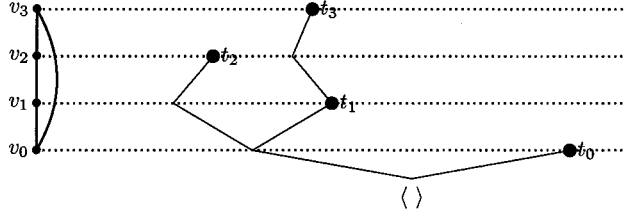


FIGURE 3. A tree coding a 4-cycle

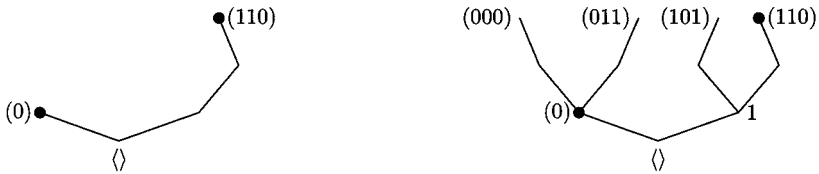
- (2) ( $f$  preserves initial segments)  $s \wedge t \subseteq u \wedge v$  if and only if  $f(s) \wedge f(t) \subseteq f(u) \wedge f(v)$ .
- (3) ( $f$  preserves relative lengths)  $|s \wedge t| < |u \wedge v|$  if and only if  $|f(s) \wedge f(t)| < |f(u) \wedge f(v)|$ .
- (4) ( $f$  preserves passing numbers) If  $|u| > |s \wedge t|$ , then  $f(u)(|f(s \wedge t)|) = u(|s \wedge t|)$ .

Whenever there is a strong similarity of  $S$  to  $T$ , we call  $S$  a *copy* of  $T$ .

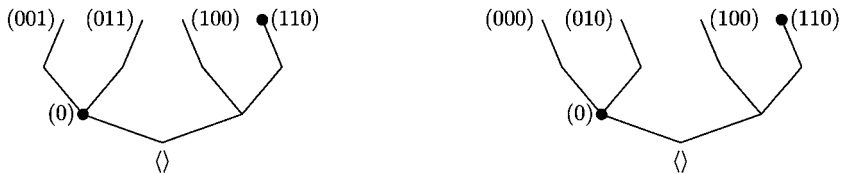
Each finite strongly diagonal  $X$  tree may be enveloped into a strong tree. Minimal envelopes will have the same number of levels as the number of meets and maximal nodes in  $X$ . The following diagram provides an example of the similarity type of a tree coding an edge, where the leftmost node is longer than the rightmost. In this example, an edge is coded by the nodes (010) and (0001), since the passing number of the sequence (0001) at length 3 = |(010)| is 1. There is only one possible envelope for this particular example of this strong similarity type coding an edge.



For some strongly diagonal trees, there can be more than one minimal envelope. The pair of nodes (0) and (110) have induced a tree with the second strong similarity type of a tree coding an edge. On the right is one envelope.



Here are two more envelopes for the same tree induced by the nodes (0) and (110) coding an edge.



The point is that given a strong similarity type of a finite strongly diagonal tree  $D$  coding a graph  $G$ , if  $k$  is the number of maximal and splitting nodes in  $D$ , then given any strong tree  $U$  isomorphic to  $2^{\leq k}$ , there is exactly one copy of  $D$  sitting inside of  $U$ . Thus, a coloring on all strongly similar copies of  $D$  inside a strong tree  $S$  may be transferred to the collection of all finite strong subtrees of  $S$  in  $\mathcal{AM}_k$ . Then Milliken's Theorem may be applied to this coloring on all copies of  $2^{\leq k}$  inside  $S$ , obtaining a strong subtree  $S' \leq S$  in which each copy of  $D$  has the same color.

The final step for Sauer's proof is to show that in any infinite strong subtree of  $2^{<\omega}$ , there is an infinite strongly diagonal tree  $\mathbb{D}$  whose terminal nodes code the Rado graph. Every finite subtree of  $\mathbb{D}$  will automatically be strongly diagonal, in particular those coding  $G$ . As there are only finitely many strong similarity types of strongly diagonal trees coding a fixed finite graph  $G$ , this provides the upper bound for the big Ramsey degree  $n(G)$ . It also provides the lower bound as each strongly diagonal type persists in any subtree of  $\mathbb{D}$  which codes the Rado graph.

Sauer's Theorem 10 was recently applied to obtain the following.

**Theorem 12** (Dobrinen, Laflamme, Sauer, [8]). *The Rado graph has the rainbow Ramsey property.*

This means that given any  $2 \leq k < \omega$ , a finite graph  $G$ , and any coloring of all copies of  $G$  in  $\mathcal{R}$  into  $\omega$  many colors, where each color appears at most  $k$  times, then there is a copy  $\mathcal{R}'$  of  $\mathcal{R}$  in which each color on a copy of  $G$  appears at most once. This result extends, with not much more work, to the larger class of binary relational simple structures. These include the random directed graph and the random tournament. It is well-known that the natural numbers have the rainbow Ramsey property, as the proof follows from the Ramsey property. On the other hand, Theorem 10 shows that the Rado graph does not have the Ramsey property when the subgraph being colored has more than one vertex. However, the finite Ramsey degrees provides enough strength to still deduce the rainbow Ramsey property for the Rado graph.

We close this section with two applications of Milliken's Theorem to colorings of finite subsets of the rationals. The rational numbers  $\mathbb{Q} = (Q, \leq_Q)$  is up to isomorphism the countable dense linear order without endpoints. The nodes in the tree  $2^{<\omega}$  may be given the following linear ordering so as to produce a copy of the rationals.

**Definition 13.** Let  $<_Q$  be the order on  $2^{<\omega}$  defined as follows. For  $s$  and  $t$  incomparable,  $s <_Q t$  if and only if  $s <_{\text{lex}} t$ . If  $s \subset t$ , then  $s <_Q t$  if and only if  $t(|s|) = 1$ ; and  $t <_Q s$  if and only if  $t(|s|) = 0$ .

With this ordering,  $(2^{<\omega}, <_Q)$  is order isomorphic to the rationals. Then Milliken's Theorem may be applied to deduce Ramsey properties of the rationals. We do not go into detail on this, but refer the interested reader to [26] for further exposition. The numbers  $t_d$  are the *tangent numbers*.

**Theorem 14** (Devlin, [1]). *Let  $d$  be a positive integer. For each coloring of  $[\mathbb{Q}]^d$  into finitely many colors, there is a subset  $Q' \subseteq \mathbb{Q}$  which is order-isomorphic to  $\mathbb{Q}$  and such that  $[Q']^d$  uses at most  $t_d$  colors. There is a coloring of  $[\mathbb{Q}]^d$  with  $t_d$  colors, none of which can be avoided by going to an order-isomorphic copy of  $\mathbb{Q}$ .*

Milliken's Theorem along with the ordering  $(2^{<\omega}, <_Q)$  isomorphic to the rationals were used to obtain the big Ramsey degrees for finite products of the rationals.

**Theorem 15** (Laver, [18]). *Let  $d$  be a positive integer. For each coloring of  $\mathbb{Q}^d$ , the product of  $d$  many copies of the rationals, into finitely many colors, there are subsets  $Q_i \subseteq \mathbb{Q}$ ,  $i < d$ , also forming sets of rationals, such that no more than  $(d+1)!$  colors occur on  $\prod_{i < d} Q_i$ . Moreover,  $(d+1)!$  is optimal.*

#### 4. THE UNIVERSAL TRIANGLE-FREE GRAPH AND RAMSEY THEORY FOR STRONG CODING TREES

The problem of whether or not countable homogeneous structures omitting a certain type of substructure can have finite big Ramsey degrees is largely open. The simplest homogeneous relational structure omitting a type is the universal triangle-free graph. A graph  $G$  is *triangle-free* if for any three vertices in  $G$ , at least one pair has no edge between them. The *universal triangle-free graph* is the triangle-free graph on  $\omega$  many vertices into which every other triangle-free graph on countably many vertices embeds. This is the analogue of the Rado graph for triangle-free graphs. The first universal

triangle-free graph was constructed by Henson in [16], which we denote as  $\mathcal{H}_3$ , in which he proved that any two countable universal triangle-free graphs are isomorphic.

There are several equivalent characterizations of  $\mathcal{H}_3$ . Let  $\mathcal{K}_3$  denote the Fraïssé class of all finite triangle-free graphs. We say that a triangle-free graph  $\mathcal{H}$  is *homogeneous* for  $\mathcal{K}_3$  if any isomorphism between two finite subgraphs of  $\mathcal{H}$  can be extended to an automorphism of  $\mathcal{H}$ .

**Theorem 16** (Henson, [16]). *Let  $\mathcal{H}$  be a triangle-free graph on  $\omega$  many vertices. The following are equivalent.*

- (1)  $\mathcal{H}$  is universal for countable triangle-free graphs.
- (2)  $\mathcal{H}$  is the Fraïssé limit of  $\mathcal{K}_3$ .
- (3)  $\mathcal{H}$  is homogeneous for  $\mathcal{K}_3$ .

In 1971, Henson proved in [16] that for any coloring of the vertices of  $\mathcal{H}_3$  into two colors, there is either a subgraph  $\mathcal{H}' \leq \mathcal{H}_3$  with all vertices in the first color, and which is isomorphic to  $\mathcal{H}_3$ ; or else there is an infinite subgraph  $\mathcal{H}' \leq \mathcal{H}_3$  in which all the vertices have the second color and into which each member of  $\mathcal{K}_3$  embeds. Fifteen years later, Komjáth and Rödl proved that vertex colorings of  $\mathcal{H}_3$  have the Ramsey property.

**Theorem 17** (Komjáth/Rödl, [17]). *For any finite coloring of the vertices  $|\mathcal{H}_3|$ , there is an  $\mathcal{H} \in \binom{\mathcal{H}_3}{\mathcal{H}_3}$  such that  $|\mathcal{H}|$  has one color.*

The next question was whether finite colorings of edges in  $\mathcal{H}_3$  could be reduced to one color on a copy of  $\mathcal{H}_3$ . In 1988, Sauer proved that this was impossible.

**Theorem 18** (Sauer, [23]). *For any finite coloring of the edges in  $\mathcal{H}_3$ , there is an  $\mathcal{H} \in \binom{\mathcal{H}_3}{\mathcal{H}_3}$  such that the edges in  $\mathcal{H}$  take on no more than two colors. Furthermore, there is a coloring on the edges in  $\mathcal{H}_3$  into two colors such that every universal triangle-free subgraph of  $\mathcal{H}_3$  has edges of both colors.*

This is in contrast to a theorem of Nešetřil and Rödl in [20] and [21] proving that the Fraïssé class of finite ordered triangle-free graphs has the Ramsey property. Sauer's result begged the question of whether his result would extend to all finite triangle-free graphs. In other words, does  $\mathcal{H}_3$  have finite big Ramsey degrees? This was recently solved by the author in [6].

**Theorem 19** (Dobrinen, [6]). *The universal triangle-free graph has finite big Ramsey degrees.*

The proof of Theorem 19 proceeds via the following steps.

- (1) Build a space of new kinds of trees, each of which codes  $\mathcal{H}_3$ . We call these *strong coding trees*. Develop a new notion of strict similarity type, which augments the notion of strong similarity type in  $2^{<\omega}$ .
- (2) Prove analogues of Halpern-Läuchli and Milliken for the collection of strong coding trees, obtaining one color per strong similarity type. These use the technique of forcing. The set-up and arguments are similar to those in Theorem 3. The coding nodes present an obstacle which must be overcome in several separate forcings; furthermore, the partial orderings are stricter than simply extension.
- (3) Develop new notion of envelope.
- (4) Apply theorems and notions from (3) and (4) to obtain a strong coding tree  $S$  with one color per strict similarity type.
- (5) Construct a diagonal subtree  $D \subseteq S$  which codes  $\mathcal{H}_3$ , and has room for the envelopes to fit in an intermediary subtree  $S'$ , where  $D \subseteq S' \subseteq S$ .
- (6) Conclude that  $\mathcal{H}_3$  has finite big Ramsey degrees.

In this article, we present the space of strong coding trees and the Halpern-Läuchli analogue, leaving the reader interested in the further steps to read [6].

One constraint for finding the finite big Ramsey degrees for  $\mathcal{H}_3$  was that, unlike the bi-embeddability between the Rado graph and the graph coded by all the nodes in  $2^{<\omega}$ , an interplay which was of fundamental importance to Sauer's proof in the previous section, there is no graph induced by a homogeneous



The trees of the form  $\mathbb{S}$  are almost what is needed. However, in order to procure the full extent of the Ramsey theorems needed, it is necessary to work with stretched versions of  $\mathbb{S}$  which are skew. This means that each level has at most one of either a node which splits or a coding node. Call the set of nodes which are either coding or splitting the set of *critical nodes*. Let  $\mathbb{T}$  denote a skewed version of  $\mathbb{S}$ , so that its coding nodes are dense in  $\mathbb{T}$  and code  $\mathcal{H}_3$ . Let  $\mathcal{T}(\mathbb{T})$  denote the collection of all subtrees of  $\mathbb{T}$  which are isomorphic to  $\mathbb{T}$ . Thus, every tree in  $\mathcal{T}(\mathbb{T})$  codes  $\mathcal{H}_3$ .

Similarly to the notation for the Milliken space in Definition 5 and following, let  $r_k(T)$  denote the first  $k$  levels of  $T$ ; thus  $r_k(T)$  contains a total of  $k$  critical nodes.  $r_{k+1}[r_k(T), T]$  denotes the set of all  $r_{k+1}(S)$ , where  $r_k(S) = r_k(T)$  and  $S$  is a subtree of  $T$  in  $\mathcal{T}(\mathbb{T})$ . The following is the analogue of the Halpern-Läuchli Theorem for strong coding trees.

**Theorem 20** (Dobrinen, [6]). *Let  $\mathbb{T}$  be a strong coding tree,  $T \in \mathcal{T}(\mathbb{T})$ ,  $k < \omega$ , and  $c$  be a coloring of  $r_{k+1}[r_k(T), T]$  into two colors. Then there is an  $S \in [r_k(T), T]$  such that all members of  $r_{k+1}[r_k(T), S]$  have the same  $c$ -color.*

The case when the maximal critical node in  $r_{k+1}(T)$  is a splitting node is significantly simpler to handle than when it is a coding node, so we present that forcing here. Enumerate the maximal nodes of  $r_{k+1}(T)$  as  $\langle s_i : i \leq d \rangle$ , and let  $s_d$  denote the splitting node. Let  $i_0$  denote the index such that  $s_{i_0} \in \{0\}^{<\omega}$ . For each  $i \neq i_0$ , let  $T_i = \{t \in T : t \supseteq s_i\}$ ; let  $T_{i_0} = 0^{<\omega}$ . Let  $L$  denote the set of levels  $l$  such that there is a splitting node at level  $l$  in  $T_d$ .

Let  $\mathbb{P}$  be the set of conditions  $p$  such that  $p$  is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \rightarrow T \upharpoonright l_p,$$

where  $\vec{\delta}_p \in [\kappa]^{<\omega}$  and  $l_p \in L$ , such that

- (i)  $p(d)$  is the splitting node extending  $s_d$  at level  $l_p$ ;
- (ii) For each  $i < d$ ,  $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \upharpoonright l_p$ .

The partial ordering on  $\mathbb{P}$  is defined as follows:  $q \leq p$  if and only if either

- (1)  $l_q = l_p$  and  $q \supseteq p$  (so also  $\vec{\delta}_q \supseteq \vec{\delta}_p$ ); or else
- (2)  $l_q > l_p$ ,  $\vec{\delta}_q \supseteq \vec{\delta}_p$ , and
  - (i)  $q(d) \supset p(d)$ , and for each  $\delta \in \vec{\delta}_p$  and  $i < d$ ,  $q(i, \delta) \supset p(i, \delta)$ ;
  - (ii) Whenever  $(\alpha_0, \dots, \alpha_{d-1})$  is a strictly increasing sequence in  $(\vec{\delta}_p)^d$  and

$$(18) \quad r_k(T) \cup \{p(i, \alpha_i) : i < d\} \cup \{p(d)\} \in r_{k+1}[r_k(T), T],$$

then also

$$(19) \quad r_k(T) \cup \{q(i, \alpha_i) : i < d\} \cup \{q(d)\} \in r_{k+1}[r_k(T), T].$$

The proof proceeds in a similar manner to that of Theorem 3, except that the name for an ultrafilter  $\mathcal{U}$  is now on  $L$  and there is much checking that certain criteria are met so that the tree being chosen by the forcing will actually again be a member of  $\mathcal{T}(\mathbb{T})$ . The actual theorem needed to obtain the finite big Ramsey degrees for  $\mathcal{H}_3$  is more involved; the statement above appears as a remark after Theorem 22 in [6]. It is presented here in the hope that the reader will see the similarity with Harrington's forcing proof.

## 5. HALPERN-LÄUCHLI THEOREM ON TREES OF UNCOUNTABLE HEIGHT AND APPLICATIONS

The Halpern-Läuchli Theorem can be extended to trees of certain uncountable heights. Here, we present some of the known results and their applications to uncountable homogeneous relational structures, as well as some open problems in this area.

The first extension of the Halpern-Läuchli Theorem to uncountable height trees is due to Shelah. In fact, he proved a strengthening of Milliken's Theorem, where level sets of size  $m$ , for a fixed positive integer  $m$ , are the objects being colored; this includes strong trees with  $k$  levels when  $m = 2^k$ , as strong trees can be recovered from the meet closures of their maximal nodes. Shelah's theorem was

strengthened to all  $m$ -sized antichains in [12]. This is the version presented here. The definition of *strong embedding* in the setting of uncountable height trees is an augmented analogue of Definition 11 for countable height trees, with the additional requirement that a fixed well-ordering  $\prec$  on the nodes in  $2^{<\kappa}$  is also preserved. A *strong subtree* of  $2^{<\kappa}$  is a subtree  $S \subseteq 2^{<\kappa}$  such that there is a set of levels  $L \subseteq \kappa$  of cardinality  $\kappa$  and the splitting nodes in  $S$  are exactly those with length in  $L$ . In particular, if  $e : 2^{<\kappa} \rightarrow 2^{<\kappa}$  is a strong embedding, then the image  $e[2^{<\kappa}]$  is a strong subtree of  $2^{<\kappa}$ . For a node  $w$  in a strong tree,  $\text{Cone}(w)$  denotes the set of all nodes in the tree extending  $w$ . The proof of the following theorem is an elaborate forcing proof, having as its base ideas from Harrington's forcing proof of Theorem 3.

**Theorem 21** (Shelah, [25]; Džamonja, Larson, and Mitchell, [12]). *Suppose that  $m < \omega$  and  $\kappa$  is a cardinal which is measurable in the generic extension obtained by adding  $\lambda$  many Cohen subsets of  $\kappa$ , where  $\lambda \rightarrow (\kappa)_{2^m}^{2^m}$ . Then for any coloring  $d$  of the  $m$ -element antichains of  $2^{<\kappa}$  into  $\sigma < \kappa$  colors, and for any well-ordering  $\prec$  of the levels of  $2^{<\kappa}$ , there is a strong embedding  $e : 2^{<\kappa} \rightarrow 2^{<\kappa}$  and a dense set of elements  $w$  such that*

- (1)  $e(s) \prec e(t)$  for all  $s \prec t$  from  $\text{Cone}(w)$ , and
- (2)  $d(e[A]) = d(e[B])$  for all  $\prec$ -similar  $m$ -element antichains  $A$  and  $B$  of  $\text{Cone}(w)$ .

A model of ZFC in which the hypothesis of the theorem holds may be obtained by starting in a model of ZFC + GCH +  $\exists a$  ( $\kappa + 2m + 2$ )-strong cardinal, as was shown by Woodin (unpublished). A cardinal  $\kappa$  is  $(\kappa + d)$ -strong if there is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $V_{\kappa+d} = M_{\kappa+d}$ .

Theorem 21 is applied in two papers of Džamonja, Larson, and Mitchell to prove that the  $\kappa$ -rationals and the  $\kappa$ -Rado graph have finite big Ramsey degrees. The  $\kappa$ -rationals,  $\mathbb{Q}_\kappa = (Q, \leq_Q)$ , is the strongly  $\kappa$ -dense linear order of size  $\kappa$ . The nodes in the tree  $2^{<\kappa}$  ordered by  $\leq_Q$  produces the  $\kappa$ -rationals, where  $\leq_Q$  is the same ordering as in Definition 13 applied to the tree  $2^{<\kappa}$ .

**Theorem 22** (Džamonja, Larson, and Mitchell, [12]). *In any model of ZFC in which  $\kappa$  is measurable after adding  $\beth_{\kappa+\omega}$  many Cohen subsets to  $\kappa$ , for any fixed positive integer  $m$ , given any coloring of of  $[\mathbb{Q}_\kappa]^m$  into less than  $\kappa$  colors, there is a subset  $Q^* \subseteq Q$  such that  $Q^* = (Q^*, \leq_Q)$  is also a strongly  $\kappa$ -dense linear order, and such that the members of  $[Q^*]^m$  take only finitely many colors. Moreover, for each strong similarity type, all members in  $[Q^*]^m$  with that similarity type have the same color.*

As each similarity type persists in every smaller copy of the  $\kappa$ -rationals, the strong similarity types provide the exact finite big Ramsey degree for colorings of  $[\mathbb{Q}_\kappa]^m$ .

The  $\kappa$ -Rado graph,  $\mathcal{R}_\kappa$ , is the random graph on  $\kappa$  many vertices. Similarly to the coding of the Rado graph using nodes in the tree  $2^{<\omega}$  in Section 3, the  $\kappa$ -Rado graph can be coded using nodes in  $2^{<\kappa}$ . Analogously to Sauer's use of Milliken's Theorem to prove that the Rado graph has finite big Ramsey degrees, Theorem 21 was an integral part in the proof obtaining finite bounds for colorings of all copies of a given finite graph inside  $\mathcal{R}_\kappa$ .

**Theorem 23** (Džamonja, Larson, and Mitchell, [13]). *In any model of ZFC with a cardinal  $\kappa$  which is measurable after adding  $\beth_{\kappa+\omega}$  many Cohen subsets of  $\kappa$ , for any finite graph  $G$  there is a finite number  $r_G^+$  such that for any coloring of the copies of  $G$  in  $\mathcal{R}_\kappa$  into less than  $\kappa$  many colors, there is a subgraph  $\mathcal{R}'_\kappa$  which is also a  $\kappa$ -Rado graph in which the copies of  $G$  take on at most  $r_G^+$  many colors.*

The number  $r_G^+$  is the number of strong similarity types of subtrees of  $2^{<\kappa}$  which code a copy of  $G$ ; recall that in the uncountable context, the fixed linear ordering on the nodes in  $2^{<\kappa}$  is a part of the description of strong similarity type. Džamonja, Larson, and Mitchell showed that for any graph  $G$  with more than two vertices, this number  $r_G^+$  is strictly greater than the Ramsey degree for the same graph  $G$  inside the Rado graph. They conclude [13] with the following question.

**Question 24** ([13]). *What is the large cardinal strength of the conclusion of Theorem 21?*

The Halpern-Läuchli Theorem for more than one tree can also be extended to uncountable height trees. Hathaway and the author considered in [7] direct analogues of the Halpern-Läuchli Theorem to more than one tree of uncountable height, also considering trees with infinite branching. A tree

$T \subseteq {}^{<\kappa}\kappa$  is a  $\kappa$ -tree if  $T$  has cardinality  $\kappa$  and every level of  $T$  has cardinality less than  $\kappa$ . We shall say that a tree  $T \subseteq {}^{<\kappa}\kappa$  is *regular* if it is a perfect  $\kappa$ -tree in which every maximal branch has cofinality  $\kappa$ . For  $\zeta < \kappa$ , let  $T(\zeta) = T \cap {}^\zeta\kappa$ .

Given a regular tree  $T \subseteq {}^{<\kappa}\kappa$ , tree  $S \subseteq T$  is called a *strong subtree* of  $T$  if  $S$  is regular and there is some set of levels  $L \subseteq \kappa$  cofinal in  $\kappa$  such that for each  $s \in S$ ,

- (1)  $s$  splits if and only if  $t$  has length  $\zeta \in L$ , and
- (2) for each  $\zeta \in L$  and  $s \in S(\zeta)$ ,  $s$  is maximally branching in  $T$ .

**Definition 25.** Let  $\delta, \sigma > 0$  be ordinals and  $\kappa$  be an infinite cardinal.  $\text{HL}(\delta, \sigma, \kappa)$  is the following statement: Given any sequence  $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$  of regular trees and a coloring

$$c : \bigcup_{\zeta < \kappa} \prod_{i < \delta} T_i(\zeta) \rightarrow \sigma,$$

there exists a sequence of trees  $\langle S_i : i < \delta \rangle$  and  $L \in [\kappa]^\kappa$  such that

- (1) each  $S_i$  is a strong subtree of  $T_i$  as witnessed by  $L \subseteq \kappa$ , and
- (2) there is some  $\sigma' < \sigma$  such that  $c$  has color  $\sigma'$  on  $\bigcup_{\zeta \in L} \prod_{i < \delta} S_i(\zeta)$ .

**Theorem 26** (Dobrinen and Hathaway, [7]). *Let  $d \geq 1$  be any finite integer and suppose that  $\kappa$  is a  $(\kappa + d)$ -strong cardinal in a model  $V$  of ZFC satisfying GCH. Then there is a forcing extension in which  $\kappa$  remains measurable and  $\text{HL}(d, \sigma, \kappa)$  holds, for all  $\sigma < \kappa$ .*

A direct lifting of the proof in Theorem 3 would yield Theorem 26, but at the expense of assuming a  $(\kappa + 2d)$ -strong cardinal. In order to bring the large cardinal strength down to a  $(\kappa + d)$ -strong cardinal, Hathaway and the author combined the method of proof in Theorem 3 with ideas from the proof of the Halpern-Läuchli Theorem in [27], using the measurability of  $\kappa$  where the methods in [27] would not directly lift. It is known that a  $(\kappa + d)$ -strong cardinal is necessary to obtain a model of ZFC in which  $\kappa$  is measurable after adding  $\kappa^{+d}$  many new Cohen subsets of  $\kappa$ . Thus, it is intriguing as to whether or not this is the actual consistency strength of the Halpern-Läuchli Theorem for  $d$  trees at a measurable cardinal, or if there is some other means for obtaining  $\text{HL}(d, \sigma, \kappa)$  at a measurable cardinal  $\kappa$ .

**Problem 27** ([7]). *Find the exact consistency strength of  $\text{HL}(d, \sigma, \kappa)$  for  $\kappa$  a measurable cardinal,  $d$  a positive integer, and  $\sigma < \kappa$ .*

We remark that a model of ZFC with a strongly inaccessible cardinal  $\kappa$  where  $\text{HL}(d, 2, \kappa)$  fails for some  $d \geq 1$  is not known at this time. We also point out that the use of  $\delta$  in the definition of  $\text{HL}(\delta, \sigma, \kappa)$ , rather than just  $d$ , is in reference to the fact that a theorem similar to Theorem 26 is proved in [7] for infinitely many trees, a result which is of interest in choiceless models of ZF. It is open whether there is a model of ZF in which the measurable cardinal remains measurable after adding the amount of Cohen subsets of  $\kappa$ . See [7] for this and other open problems.

Various weaker forms of the Halpern-Läuchli Theorem were also investigated in [7]. The *somewhere dense Halpern-Läuchli Theorem*,  $\text{SDHL}(\delta, \sigma, \kappa)$ , is the version where it is only required to find levels  $l < l' < \kappa$ , nodes  $t_i \in T_i(l)$ , and sets  $S_i \subseteq T_i(l')$  such that each immediate successor of  $t_i$  in  $T_i$  is extended by a unique member of  $S_i$ , and all sequences in  $\prod_{i < \delta} S_i$  have the same color.

**Theorem 28** (Dobrinen and Hathaway, [7]). *For a weakly compact cardinal  $\kappa$  and ordinals  $0 < \delta, \sigma < \kappa$ ,  $\text{SDHL}(\delta, \sigma, \kappa)$  holds if and only if  $\text{HL}(\delta, \sigma, \kappa)$ .*

Very recently, Zhang extended Laver's Theorem 15 to the uncountable setting. First, he proved a strengthened version of Theorem 26, though using a stronger large cardinal hypothesis. Then he applied it to prove the following.

**Theorem 29** (Zhang, [28]). *Suppose that  $\kappa$  is a cardinal which is measurable after forcing to add  $\lambda$  many Cohen subsets of  $\kappa$ , where  $\lambda$  satisfies the partition relation  $\lambda \rightarrow (\kappa)_{2^d}^{2^d}$ . Then for any coloring of the product of  $d$  many copies of  $\mathbb{Q}_\kappa$  into less than  $\kappa$  many colors, there are copies  $Q_i$ ,  $i < d$ , of  $\mathbb{Q}_\kappa$  such that the coloring takes on at most  $(d + 1)!$  colors on  $\prod_{i < d} Q_i$ . Moreover,  $(d + 1)!$  is optimal.*



As discussed previously, it is consistent with a  $(\kappa + 2d)$ -strong cardinal to have a model where  $\kappa$  satisfies the hypothesis of Theorem 29. Zhang asked whether the conclusion of this theorem is a consequence of some large cardinal hypothesis. The author asks the following possibly easier question.

**Question 30.** *Is a  $(\kappa + d)$ -strong cardinal sufficient to produce a model in which the conclusion of Theorem 29 holds?*

In [7], it was proved in ZFC that  $\text{HL}(1, k, \kappa)$  holds for each positive integer  $k$  and each weakly compact cardinal  $\kappa$ . This was recently extended by Zhang in [28] to colorings into less than  $\kappa$  many colors; moreover, he proved this for a stronger asymmetric version. Furthermore, Zhang showed that relative to the existence of a measurable cardinal, it is possible to have  $\text{HL}(1, \delta, \kappa)$  hold where  $\kappa$  is a strongly inaccessible cardinal which is not weakly compact. Still, the following basic question remains open.

**Question 31.** *Is it true in ZFC that if  $\kappa$  is strongly inaccessible, then  $\text{HL}(1, \delta, \kappa)$  holds, for some (or all)  $\delta$  with  $2 \leq \delta < \kappa$ ?*

Mapping out all the implications between the various forms of the Halpern-Läuchli Theorem at uncountable cardinals, their applications to homogeneous relational structures, and their relative consistency strengths is an area ripe for further exploration.

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